Supplement to "Some Aspects of Minimally Relativistic Newtonian Gravity"

Kamal K. Nandi

University of North Bengal, Department of Mathematics, Darjeeling (W.B.), Pin-734430, India

This supplement to Z. Naturforsch. **46a**, 1026 (1991) provides additional material that should be combined with the contents of that paper. The combined content then provides a model of gravitation which is complete in essential details. Its main features are that it (i) predicts all the local GR tests accurately but (ii) does *not* exhibit Schwarzschild singularity, (iii) allows the wellknown Birkhoff's theorem to hold and (iv) has well defined conservation laws in virtue of the maximal group of motions admissible on the flat background spacetime.

I. Introduction

This supplement contains clarifications, improvements and some corrections on [1]. These changes will become self evident as we go on. The present paper together with the original one is expected to complete the essentials of the proposed "field" theory of gravitation. The present sections are to be regarded as applicable to the corresponding sections of the original paper. Equations of the supplement are referred to as (A), (B), ... while the numbered equations refer to those in the previous paper.

Let us summarize the main features of the two papers put together. The present model is distinct from Einstein's General Relativity (GR) and it (i) predicts all the local tests of GR accurately, (ii) allows Birkhoff's theorem to hold good, (iii) has well defined conservation laws but (iv) does not exhibit Schwarzschild singularity.

II. The Field Equations

The energy momentum tensor of matter $\varrho_{\mu\nu}$ in the field equation (1) should be regarded not as one from Special Relativity but, in virtue of the principle of equivalence, as one containing $h_{\mu\nu}$ and $\eta_{\mu\nu}$ only in the combination $a_{\mu\nu}$. This tensor was defined in (3) as a "field-flatmetric" coupling: $a_{\mu\nu} = \eta_{\mu\nu} - (h_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}h)$. The redefinition of $\varrho_{\mu\nu}$ does not, however, affect the exterior Schwarzschild problem since, anyway, in empty space $\varrho_{\mu\nu} \equiv 0$.

III. The Equations of Motion and Weak Field Limit

It was argued in the previous paper that the numerical value of the perihelic shift of a planet (say, Mercury) fixes the role of $a_{\mu\nu}$ as the Riemannian metric. However, the argument could be valid *only if* we had known a priori the value of the unknown parameter $\varkappa$ appearing in the expression for $a_{\mu\nu}$. Hence, we choose to reserve the value of the perihelic shift for fixing the value of $\varkappa$ and identify $a_{\mu\nu}$ with the Riemannian metric *in complete formal analogy* with what is done elsewhere. For example, Grishchuk, Petrov, Popova, and Zel'dovich [2] reformulate GR as a "field" theory on a flat background but the equivalence of the two formulations is revealed *only* after the tensor resulting from the coupling similar to (3) is formally identified with the Riemannian metric.

The task for us is to find the solution for the potentials $h_{\mu\nu}$ (and hence $a_{\mu\nu}$) from the field equations of this model. The equations of motion can either be the Hamilton-Jacobi equation as before or the geodesic equation. Both equations produce the same results, of course.

IV. Solution of the Schwarzschild Problem

With $a_{\mu\nu} = (a_{00}, a_{ii})$ in (11) and (12), we have explicitly,

$$ds^2 = -a_{\mu\nu}dx^\mu dx^\nu = \left[1 + \frac{1}{2\varkappa}\ln\left(1 - \frac{4\varkappa M_0}{r}\right)\right]dt^2$$
$$- \left[1 - \frac{1}{2\varkappa}\ln\left(1 - \frac{4\varkappa M_0}{r}\right)\right]$$
$$\cdot (dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta d\varphi^2).$$

Expanding,

$$ds^2 = \left(1 - \frac{2 M_0}{r} - \frac{4 \varkappa M_0^2}{r^2} + \ldots\right) dt^2 \tag{A}$$

$$- \left(1 + \frac{2 M_0}{r} + \ldots\right)[dr^2 + r^2 d\theta^2 + r^2 \sin^2 \theta \, d\varphi^2].$$

Now, let us display the Eddington-Robertson expansion corresponding to the Schwarzschild problem [3]:

$$ds^2 = \left(1 - 2\alpha\frac{M_0}{r} + 2\beta\frac{M_0^2}{r^2} + \ldots\right) dt^2 \tag{B}$$

$$- \left(1 + 2\gamma\frac{M_0}{r} + \ldots\right)[dr^2 + r^2 d\theta^2 + r^2 \sin^2 \theta \, d\varphi^2],$$

where α, β, γ are constants. Recall that all the predictions of GR for this problem can be neatly summarized as

$$\alpha = \beta = \gamma = 1.$$

Comparing (A) and (B) above, we already see that $\alpha = \gamma = 1$, and if we choose $\varkappa = -\frac{1}{2}$ in (A), we also have $\beta = 1$. It is to be remembered that, of all the Schwarzschild tests in GR, only the value of the perihelic precession is sensitive to terms M_0^2 in the expansion for g_{00}. In view of this, we can say that the numerical value of $\varkappa$ is fixed by this precession event. Clearly, one can now see without any detailed calculations, that the proposed model with $\varkappa = -\frac{1}{2}$ also predicts precisely the same results as those of GR in this problem.

Obviously, the consideration of other values, viz. $\varkappa = 0$ or $-\frac{1}{8}$, are now out of question. The former value was contemplated as a result of an oversight. In (18), the factor multiplying $6 m_0^2 M_0^2$ should read $(\frac{4}{3} + \frac{2}{3}\varkappa)$ instead of $(1 + \frac{2}{3}\varkappa)$. The other value arose out of a comparison between two approximate expressions, (41) and (44). Of course, now we see that we can determine x exactly.

V. Time Dependent Solution

The conclusion that Birkhoff's theorem does not hold in our model is incorrect. In fact, it *does* hold. To see this, one has only to proceed just a few steps as follows:

We are looking for a real solution of (20) since Birkhoff's theorem does not involve complex quantities. The general solution of (21), which is finite every-

where except at $r = 0$, is

$$e^{\varkappa h_{00}} = \frac{A}{r}[F(r - c\,t) + G(r + c\,t)], \tag{C}$$

where A is a constant, F and G are arbitrary functions. We retain only the retarded solution and discard the other so that, for spherically symmetric waves, we have finally

$$e^{\varkappa h_{00}} = \frac{A}{r} e^{i(kr - \omega t)}, \quad \frac{\omega}{k} = c.$$

This implies, taking the principal value,

$$h_{00}(r, t) = \frac{1}{\varkappa} \ln \frac{A}{r} + \frac{i}{\varkappa}(kr - \omega t). \tag{D}$$

It is evident that the time dependence has become imaginary. This leads to the conclusion that Birkhoff's theorem holds. The original version of this theorem states that any space-time with spherical symmetry must be static. Our $a_{\mu\nu}$, too, satisfies this condition.

The real part of the solution is time independent:

$$\tilde{h}_{00}(r) = \operatorname{Re}[h_{00}(r, t)] = \frac{1}{\varkappa} \ln(A/r),$$

which is itself a solution of (20). It is now a formal matter to rewrite the asymptotic and weak field conditions in terms of $\tilde{h}_{00}$. They are utilized such that the solution is

$$\tilde{h}_{00}(r) = \frac{1}{\varkappa} \ln\left(1 - \frac{4\varkappa M_0}{r}\right) \equiv h_{00}(r) \tag{E}$$

and, in the end, we have precisely the same expressions for a_{00} and a_{ii} as in (11) and (12).

The well known corollary to Birkhoff's theorem also holds. Consider a spherical cavity. Inside, at $r = 0$, we have no matter and consequently the point falls in the empty space. Therefore the metric $a_{\mu\nu}$ or the field $h_{\mu\nu}$ must be regular there. This requires that $M_0 = 0$, so that $a_{\mu\nu} = (-1, 1, r^2, r^2 \sin^2 \theta)$, a flat metric.

VI. Conservation Laws

The conservation laws have a well defined meaning in the present model in that the numerical value of the physically measurable quantities such as energy and momentum do not depend on the choice of coordinates.

With the recalculated value of $\varkappa = -\frac{1}{2}$, it may be noted, we get out of the Ashbrook-Dicke-Goldenberg controversy. And finally, there is *no* singularity at $r = M_0/2$. There is, however, a mathematical singularity in both potentials and metric components at $r = -2 M_0$. But that is the same as saying that there is no physical singularity in our model, at least in the Schwarzschild problem.

It is hoped that the model proposed in our two papers, read in conjunction, should be of some use since it illustrates in its formulation apart from the GR predictions, also the underlying physics.

[1] K. N. Nandi, Z. Naturforsch. **46 a,** 1026 (1991).
[2] L. P. Grishchuk, A. N. Petrov, and A. D. Popova, Comm. Math. Phys. **94,** 379 (1984); L. P. Grishchuk and Ya. B. Zel'dovich, Sov. Phys. Usp. **29 (8),** 780 (1986).

[3] S. Weinberg, Gravitation and Cosmology, Wiley, New York 1972.